SHSAT Lessons 7: Probability and Combinatorics

In this lesson:

Basic Probability: Outcomes. Sample Space.
Combinatorics: Permutations. Combinations. Dirichlet's principle.

What you must know

Definitions

Sample Space is the list of all possible outcomes (or events) that can occur.
Probability is the ratio (or fraction) of favorable outcomes to the total numbers of possible outcomes. Typically, the chance that an event or series of events will occur is expressed on a scale from 0 (impossible) to 1 (certainty) or as an equivalent percentage from 0 to 100%.
Dirichlet's box principle states that if n items are put into m boxes and $n > m$ (which is to say that the number of items is greater than number of boxes), then at least one box will contain more than one item. For example, if you have 5 rabbits and you put them in 4 cages, it's guaranteed that one cage would contain more than one rabbit.

Sample Problems

1. You have four coins: a quarter, a dime, a nickel, and a penny. You are making different fractions with the cent value of one coin in the numerator and the cent value of another coin in the denominator. How many fractions with distinct values can you make?

Solution

Write out an organized list, starting with all fractions with 1 in the numerator (and listing them by the value in the denominator), and so on.

$$\frac{1}{5}, \frac{1}{10}, \frac{1}{25}, \frac{5}{1}, \frac{5}{10}, \frac{5}{25}, \frac{10}{1}, \frac{10}{5}, \frac{10}{25}, \frac{25}{1}, \frac{25}{5}, \frac{25}{10}.$$

There are $3 \times 4 = 12$ outcomes, but we have 2 pairs of fractions with equal values: $\frac{1}{5} = \frac{5}{25}$ and $\frac{25}{5} = \frac{5}{1}$, so each of them only counts once. Therefore, the answer is **10**.

2. If you randomly pick a day in January, what is the greatest possible probability that the chosen day is Sunday?

Solution

There can be 4 or 5 Sundays in January (depends on what day of the week January 1 is). Therefore, the probability of randomly picking a Sunday is either $\frac{5}{31}$ or $\frac{4}{31}$. Since $\frac{5}{31} > \frac{4}{31}$, the answer is $\frac{5}{31}$.

3. A room contains 10 cats and 15 small toys. Each cat is playing with at least one toy, but with no more than 2 toys. If all toys are being played with, how many cats are playing with 2 toys?

Solution

If each cat is given 1 toys, there are 5 toys left. Since all toys are in play, these five toys are being played with. But since each cat cannot have more than 2 toys, exactly **5 cats** have 2 toys.

4. If $|x| = 2$, and $|y| = 3$, and the signs of x and y are randomly selected, what is the probability that $|x + y| = 1$?

Solution

Since x could be 2 or –2, and y could be 3 or –3, the sum of x and y could be 5, –5, 1, or –1, which means that $|x + y|$ could be 5 or 1.

Therefore, the probability of getting 1 is 1 out of 2, or $\frac{1}{2}$.

5. If the probability of picking a white marble from a bag is $\frac{3}{4}$, and there are 12 white marbles in the bag, how many marbles in the bag are not white?

Solution

There are 12 white marbles in the bag and they constitute $\frac{3}{4}$ of all marbles. Therefore, the number of non-white marbles is $1 - \frac{3}{4} = \frac{1}{4}$ of the total. Since we know that $\frac{3}{4}$ of the total is 12, $\frac{1}{4}$ of the total is 3 times less, or $12 \div 3 = 4$.

Practice Problems

1. There are 4 red and 5 yellow marbles in the bag. If you pick 2 marbles without looking, what is the probability that both marbles you picked are red?

   A. $\frac{1}{3}$
   B. $\frac{1}{4}$
   C. $\frac{1}{6}$
   D. $\frac{1}{9}$

2. The locker's combination code consists of one letter and 3 digits. How many possible different codes exist, if the letter can be A or C and must come first, and the digits can be repeated?

   E. 600
   F. 1200
   G. 1500
   H. 2000

3. 2 boys and 2 girls are sitting around a square table. Each seat is numbered 1, 2, 3, and 4. How many different arrangements of the 4 children are possible?

   A. 4
   B. 16
   C. 24
   D. 32

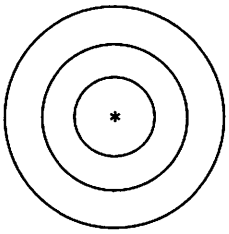
4. There are 25 students in a class. What is the smallest number of children that are guaranteed to have their birthdays in the same month?

   E. 0
   F. 1
   G. 2
   H. 3

5. There are 3 small cubes in the bag: red, blue and green. You are picking cubes from the bag one by one. What is the chance that you'll pick the red cube last?

   A. $\frac{1}{3}$
   B. $\frac{1}{5}$
   C. $\frac{1}{9}$
   D. $\frac{2}{9}$

6. The target consists of three concentric circles with the radii 1, 2, and 3 units respectively. If a dart hits the target randomly, what is the probability of hitting the innermost circle on the first shot?



E. $\frac{1}{3}$
F. $\frac{2}{3}$
G. $\frac{1}{5}$
H. $\frac{1}{9}$

---

7. If you write down all positive integers from 1 to 50, what is the total number of times that 3 and 7 will appear on paper?

A. 20
B. 18
C. 8
D. 7

---

8. Find the largest factor of 1568 that is not divisible by 7.

E. 254
F. 392
G. 32
H. 98

---

9. Two boys and three girls want to sit in a row of 5 chairs. How many possible arrangements exist, if a boy always sits in the first and the last chair?

A. 24
B. 12
C. 6
D. 2

---

10. How many three-digit positive numbers have only odd digits?

E. 125
F. 60
G. 28
H. 24

---

11. What is the unit (ones) digit of the number 3^{13} ?

A. 1
B. 3
C. 9
D. 8

---

12. Which integer of the ones below has the largest number of prime factors?

E. 143
F. 140
G. 130
H. 128

13. How many more prime numbers are between 10 and 30 than between 10 and 20?

A. 1
B. 2
C. 3
D. 4

14. There are 20 balls, numbered from 1 to 20, in the bag. If you pick one ball from the bag, what is the probability of picking a prime-numbered ball?

E. $\frac{1}{3}$
F. $\frac{2}{3}$
G. $\frac{2}{5}$
H. $\frac{9}{20}$

15. What is the probability of picking 2 red marbles from the jar that contains 2 blue and 2 red marbles?

A. $\frac{1}{4}$
B. $\frac{3}{4}$
C. $\frac{1}{6}$
D. $\frac{1}{2}$

16. There are 14 people in the room. What is the probability that at least 2 people have their birthday in the same month?

E. $\frac{1}{12}$
F. $\frac{1}{7}$
G. 1
H. 0

17. Seven birds are placed into 3 cages. What is the smallest possible number of birds in the cage with the largest number of birds?

A. 1
B. 2
C. 3
D. 4

18. Repeating fraction $\frac{1}{7}$ was rounded to a random length in its decimal expansion. What is the probability that the last digit in the rounded value is 3?

E. $\frac{1}{8}$
F. $\frac{1}{6}$
G. 1
H. 0

19. Balls with letters that form the word MATHEMATICS are placed in a bowl. What is the probability of first picking a ball with a letter other than "C" and then picking one with a "C"?

A. $\frac{3}{4}$
B. $\frac{1}{12}$
C. $\frac{1}{10}$
D. $\frac{1}{11}$

20. In a group of students, 16 got an A in Math, 14 got an A in Science, and 5 got both an A in Math and an A in Science. If a student is selected at random, find the probability, expressed as a percentage, that the student has an A in Math and an A in Science.

E. 20%
F. 16%
G. 14%
H. 12%

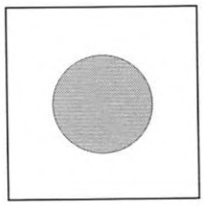
21. If you randomly select a two-digit integer, what is the probability that it is not a multiple of 5?

A. $\frac{64}{81}$
B. $\frac{4}{5}$
C. $\frac{11}{18}$
D. $\frac{11}{81}$

22. If the letters in the word LATE are randomly rearranged, what is the probability that the resulting word will be TALE?

E. $\frac{1}{7}$
F. $\frac{1}{12}$
G. $\frac{1}{24}$
H. 0

23. The area of a circle inside a square is equal to $\frac{1}{3}$ of the area of the square. What is the probability that the point randomly selected inside the square will be outside of the circle?



A. $\frac{1}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{8}$

D. $\frac{1}{2}$

24. Mr. Jones has 6 coworkers: 2 men and 4 women. How many different groups of 2 people could be selected for a project if a group must have 1 man and 1 woman?

E. 8

F. 32

G. 24

H. 12

25. You are given two sets, X = {2, 5, 7, 9} and Y = {1, 3, 4, 8}. A number is randomly selected from each set. What is the probability that the sum of the two selected numbers is 10?

A. $\frac{1}{16}$

B. $\frac{2}{3}$

C. $\frac{3}{16}$

D. $\frac{1}{3}$